OPTIMIZATION OF THE HEATING MODE IN THE VULCANIZATION OF RUBBER PRODUCTS

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Optimality criteria are selected for the vulcanization process with external heating of the rubber product. The variational isoperimetric problem is formulated here with appropriate limits. The optimum in this sense heating mode is determined by numerical integration of the Euler equation on a digital computer.

The manufacture of modern technical-grade rubber products such as, for example, automobile tires, involves very stringent requirements concerning the heating modes during vulcanization, which, in the final analysis, determine the quality and the cost of these products. We will be concerned here with the problems of producing homogeneous vulcanites within a minimum process time, which is especially important in the vulcanization of products with a large thermal mass.

The problem of heat conduction in an infinitely large plate with boundary conditions of the first kind will be expressed in dimensionless form:

$$\frac{\partial \vartheta (X,\ \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{\partial^2 \vartheta (X,\ \mathrm{Fo})}{\partial X^2} \quad (0 < X < 1,\ 0 \leqslant \mathrm{Fo} < \infty), \tag{1}$$

$$\vartheta (X,\ 0) = 1 \quad (0 < X < 1), \tag{2}$$

$$\vartheta (0,\ \mathrm{Fo}) = \psi_1 (\mathrm{Fo}) \quad (0 \leqslant \mathrm{Fo} < \infty), \tag{3}$$

$$\vartheta (1,\ \mathrm{Fo}) = \psi_2 (\mathrm{Fo}) \quad (0 \leqslant \mathrm{Fo} < \infty). \tag{4}$$

The conversion level of a resion mix is determined on the basis of the "equivalent time" $\mathrm{Fo_e}$ according to the equation

$$\mathrm{Fo_e}(X,\ \mathrm{Fo}) = \exp\left(\frac{E}{RT_e}\right) \int\limits_0^{\mathrm{Fo}} \exp\left[-\frac{E}{RT_0 \vartheta (X,\ \mathrm{Fo}^*)}\right] d\mathrm{Fo}^*. \tag{5}$$

We seek a relation $\psi_1 = \psi_2 = \psi$ so that, when $\mathrm{Fo} = \mathrm{Fo}^k$, the difference

$$J_0 = \mathrm{Fo_e}(0,\ \mathrm{Fo}^k) - \mathrm{Fo_e}\left(\frac{1}{2},\ \mathrm{Fo}^k\right) \tag{6}$$

be minimum under the conditions

$$\mathrm{Fo_e}\left(\frac{1}{2},\ \mathrm{Fo}^k\right) = A, \qquad \psi (\mathrm{Fo}) \geqslant \psi_0, \tag{7}$$

with the constant A determined by the minimum value of $\mathrm{Fo_e}$ at which the vulcanization level is still adequate, with ψ_0 denoting the temperature threshold determined on the basis of technological considerations.

The solution to the heat conduction problem (1)-(4) for $\psi_1 = \psi_2 = \psi$ will then be [2]

$$\vartheta (X,\ \mathrm{Fo}) = \psi (\mathrm{Fo}) + [1 - \psi (0)] \sum_{n=1}^{\infty} \frac{4}{\pi (2n-1)} \sin [\pi (2n-1) X]$$

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$$\times \exp\{-[\pi(2n-1)]^2 \mathrm{Fo}\} - \int_0^{\mathrm{Fo}} \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)} \sin[\pi(2n-1)X]$$

$$\times \exp\{-[\pi(2n-1)]^2 (\mathrm{Fo}-\mathrm{Fo}^*)\}\, \psi'(\mathrm{Fo}^*)\, d\mathrm{Fo}^*. \tag{8}$$

To the first approximation,

$$\vartheta\left(\frac{1}{2},\ \mathrm{Fo}\right) = \exp(-\pi^2 \mathrm{Fo}) \left[1 + \pi^2 \int_0^{\mathrm{Fo}} \psi(\mathrm{Fo}^*) \exp(\pi^2 \mathrm{Fo}^*)\, d\mathrm{Fo}^*\right]. \tag{9}$$

We now introduce the function

$$\varphi(\mathrm{Fo}) = \int_0^{\mathrm{Fo}} \psi(\mathrm{Fo}^*) \exp(\pi^2 \mathrm{Fo}^*)\, d\mathrm{Fo}^*, \tag{10}$$

$$\varphi'(\mathrm{Fo}) = \psi(\mathrm{Fo}) \exp(\pi^2 \mathrm{Fo}). \tag{11}$$

Then functional (6) becomes

$$J_0[\varphi] = \exp\left(\frac{E}{RT_e}\right) \int_0^{\mathrm{Fo}^k} \left\{\exp\left[-\frac{E\exp(\pi^2 \mathrm{Fo})}{RT_0\varphi'(\mathrm{Fo})}\right]\right.$$

$$\left. - \exp\left[-\frac{E\exp(\pi^2\mathrm{Fo})}{RT_0[1+\pi^2\varphi(\mathrm{Fo})]}\right]\right\} d\mathrm{Fo}. \tag{12}$$

Thus, the problem has been reduced to finding the function $\varphi(\mathrm{Fo})$ which will minimize functional (12)

$$J_0[\varphi] = \int_0^{\mathrm{Fo}^k} F(\mathrm{Fo},\ \varphi,\ \varphi')\, d\mathrm{Fo}$$

under the conditions

$$\int_0^{\mathrm{Fo}^k} G(\mathrm{Fo},\ \varphi)\, d\mathrm{Fo} = A, \tag{7a}$$

$$\varphi'(\mathrm{Fo}) \geqslant \psi_0 \exp(\pi^2 \mathrm{Fo}),$$

$$\varphi(0) = 0, \tag{13}$$

$$\varphi(\mathrm{Fo}^k) = \varphi^k. \tag{14}$$

Condition (14) corresponds to the required temperature $\vartheta(1/2,\ \mathrm{Fo}^k)$.

Let us dwell on the limitation (7a). Without this limitation, a variational isoperimetric problem [3] is formulated whose solution must be the extremal of the functional

$$J_1[\varphi] = \int_0^{\mathrm{Fo}^k} (F + \lambda G)\, d\mathrm{Fo} \tag{15}$$

under conditions (7), (13), and (14). Here λ is a certain constant determined from condition (7).

The Euler equation for functional (15) is

$$\frac{1-\lambda}{(1+\pi^2\varphi)^2} \exp\left[-\frac{E\exp(\pi^2\mathrm{Fo})}{RT_0(1+\pi^2\varphi)}\right] + \frac{1}{(\varphi')^2} \exp\left[-\frac{E\exp(\pi^2\mathrm{Fo})}{RT_0\varphi'}\right]$$

$$\times \left\{1 - \frac{E\exp(\pi^2\mathrm{Fo})}{RT_0\varphi'} - \frac{2\varphi''}{\pi^2\varphi'}\left[1 - \frac{E\exp(\pi^2\mathrm{Fo})}{RT_0 2\varphi'}\right]\right\} = 0. \tag{16}$$

We note the following.

Note 1. The necessary Legendre condition for a minimum of functional (15)

$$(F + \lambda G)_{\varphi'\varphi'} > 0$$

is satisfied at any values of $\mathrm{Fo} \geq 0$:

$$\frac{E}{RT_0(\varphi')^4} \exp\left[\pi^2\mathrm{Fo} - \frac{E\exp(\pi^2\mathrm{Fo})}{RT_0\varphi'}\right]\left[\frac{E\exp(\pi^2\mathrm{Fo})}{RT_0} - 2\varphi'\right] > 0,$$

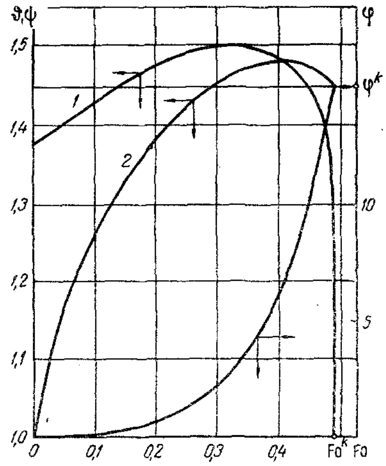

Fig. 1

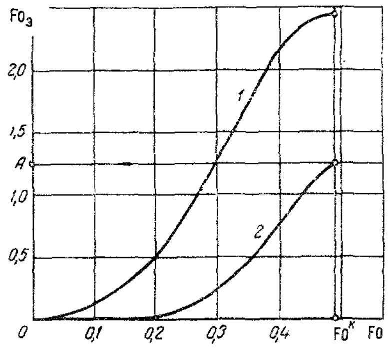

Fig. 2

Fig. 1. Variation in the relative temperature at surface points ψ(Fo) (1) and at the center of a plate ϑ(1/2, Fo) (2), and the minimizing function φ (Fo).

Fig. 2. Variation of function Fo_e(Fo), in the optimum mode, at surface points Fo_e(0, Fo) (1) and at the center of a plate Fo_e(1/2, Fo) (2).

$E/RT_0 > 2\psi$ in accordance with condition (7), which, in fact, limits ψ(Fo) from above.

Note 2. The form of Eq. (16) (Euler) indicates that the expression inside the square brackets will retain its sign at any value of Fo ≥ 0. Consequently,

$$\psi'(\mathrm{Fo}) > \frac{\pi^2\psi^2(\mathrm{Fo})}{\dfrac{E}{RT_0} - 2\psi(\mathrm{Fo})} \quad \text{for } 1-\lambda < 0 \tag{17}$$

and

$$\psi'(\mathrm{Fo}) \leqslant \frac{\pi^2\psi^2(\mathrm{Fo})}{\dfrac{E}{RT_0} - 2\psi(\mathrm{Fo})} \quad \text{for } 1-\lambda \geqslant 0. \tag{18}$$

Note 3. A preliminary analysis of the solution to the Euler equation, made with the aid of a digital computer, indicates that only from a certain value Fo = Fo** on is condition (7a) violated at all values of Fo > Fo**.

In this case the solution to the problem must be sought on the basis of the generalized Euler theorem [4]. Condition (7a) can be represented in the form

$$\begin{gathered} \varphi(\mathrm{Fo}) \geqslant 0 \quad (0 \leqslant \mathrm{Fo} < \mathrm{Fo}^{**}), \\ \varphi(\mathrm{Fo}) \geqslant \frac{\psi_0}{\pi^2}\left[\exp(\pi^2\mathrm{Fo}) - \exp(\pi^2\mathrm{Fo}^{**})\right] + \varphi(\mathrm{Fo}^{**}) \quad (\mathrm{Fo}^{**} \leqslant \mathrm{Fo} \leqslant \mathrm{Fo}^k). \end{gathered} \tag{7b}$$

Point φ^k now lies on the boundary of the permissible region of extremals. The boundary condition (14) is superseded accordingly. The corresponding constant in the solution to the Euler equation (16) and the corresponding value of Fo** can be found from the system of equations

$$\begin{gathered} \varphi'(\mathrm{Fo}^{**}, c_2) = \psi_0 \exp(\pi^2\mathrm{Fo}^{**}), \\ \varphi(\mathrm{Fo}^{**}, c_2) = \frac{\psi_0}{\pi^2}\left[\exp(\pi^2\mathrm{Fo}^{**}) - \exp(\pi^2\mathrm{Fo}^k)\right] + \varphi^k. \end{gathered} \tag{19}$$

Thus, the solution will consist of extremal segments for $0 \leq \mathrm{Fo} \leq \mathrm{Fo}^{**}$ and the boundary of the permissible region defined by conditions (7b). The extremal intersects this boundary at point Fo**, and point φ^k lies on this boundary. In specific cases Fo** = Fo^k or Fo** lies outside the [0, Fo^k] interval where functional (15) is defined.

Note 4. Function φ(Fo), which minimizes functional (15) under conditions (7), (7b), (13), and (14), maximizes the functional

$$J_2 = \mathrm{Fo_e}\left(\frac{1}{2},\ \mathrm{Fo}^k\right)$$

under the conditions

$$\mathrm{Fo_e}(0,\ \mathrm{Fo}^k) = B$$

and (7b), (13), (14), by virtue of the equivalency of problems regarding B which corresponds to $\mathrm{Fo_e}(0,\ \mathrm{Fo^k})$ in the functional problem.

The Euler equation (16) was integrated numerically on a digital computer by the Runge—Kutta method. Function $\varphi(\mathrm{Fo^k}) = \varphi^k$ was searched, with an error $\varphi(\mathrm{Fo^k})-\varphi^k$ taken into account, for various values of λ and with condition (7) satisfied. The computations were made for the following boundary conditions and values of physical parameters: $\varphi(\mathrm{Fo^k}) = 15$, $\mathrm{Fo^k} = 0.4925$, $\varphi(1/2,\ \mathrm{Fo^k}) = 1.45$, $A = 1.25$, E/R, $T_0 = 40$, $\psi_0 = 1$. The optimum control for heating a plate, in the sense of the problem as formulated here, was obtained with $\lambda = 1$. $\mathrm{Fo_e}(0,\ \mathrm{Fo^k}) = 2.443$.

For comparison, in accordance with note 4, a near-optimum mode was selected with a constant temperature $\psi = 1.466$. The initial and the boundary conditions remain almost unchanged. With $\mathrm{Fo_e}(0,\ \mathrm{Fo^k}) = 2.443$, the result was $\mathrm{Fo_e}(1/2,\ \mathrm{Fo^k}) = 0.945$, as compared with 1.250 in the optimum mode.

We note that the exact solution (8) to the heat conduction problem at $X = 1/2$ is approximated by expression (9) adequately enough with respect to the sought boundary conditions. This, the error in the described example did not exceed 3% at temperature levels appropriate for the given problem.

NOTATION

$\vartheta(X,\ \mathrm{Fo}) = T(X,\ \mathrm{Fo})/T_0$ is the relative temperature at point $X = x/l$ at the instant of time $\mathrm{Fo} = a\tau/l^2$;
T is the temperature, °K;
T_0 is the initial plate temperature;
x is the space coordinate;
τ is the time coordinate;
l is the thickness;
a is the thermal diffusivity;
ψ is the relative temperature at the plate surface;
$\mathrm{Fo_e} = a\tau_e/l^2$ is the dimensionless "equivalent time";
τ_e is the equivalent time.
E is the activation energy of the vulcanizing reaction;
R is the universal gas constant;
T_e is the reference temperature (°K) at which processing for a length of time τ_e results in the same degree of vulcanizing as processing at temperature T for the length of time τ.

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